Appendix B

Decomposition with Respect to a Basis Using a Scalar Product

The purpose of this appendix is showing how, in a finite-dimensional vector space V with a basis, a scalar product $< \cdot, \cdot >$ can be used to decompose an arbitrary element u with respect to that basis.

Suppose the basis elements are $\{v_1, \ldots, v_n\}$. We are looking for numbers $a_1, \ldots, a_n$ such that

$$u = \sum_{i=1}^{n} a_i v_i \tag{B.1}$$

Since $\{v_1, \ldots, v_n\}$ are by assumption a basis, such a decomposition exists and is unique, by definition of a basis. The question is how to find the coefficients a_i effectively. This works as follows.

Theorem 1 *Let* $b_j =< u, v_j >$, $B_{ij} =< v_i, v_j >$ *and* $A = B^{-1}$. *Then* $a_i = \sum_j A_{ij} b_j$.

Proof: Consider the decomposition (B.1). Taking scalar products of both sides with each basis vector v_j gives us

$$< u, v_j > \;=\; \sum_{i=1}^{n} a_i < v_i, v_j > \quad \Leftrightarrow \tag{B.2}$$

$$b_j \;=\; \sum_{i=1}^{n} a_i B_{ij} \quad \Leftrightarrow \tag{B.3}$$

$$b_j \;=\; \sum_{i=1}^{n} B_{ji} a_i \tag{B.4}$$

where we have used $B_{ij} =< v_i, v_j >=< v_j, v_i >= B_{ji}$. If we define b to be the column vector made of the elements b_i, and a to be the column vector made of the elemens a_i, then (B.4) can be written in matrix notation as

$$Ba \;=\; b \quad \Leftrightarrow \tag{B.5}$$

$$a \;=\; B^{-1} b \tag{B.6}$$

or writing out the last equality element by element,

$$a_i = \sum_j A_{ij} b_j \tag{B.7}$$

Q.E.D.

The beauty of this result is that it holds in *any* finite-dimensional vector space V that has a scalar product. u and v_i can be anything, as long as they are something we can add up to each other and multiply by real numbers. For example, in our case they were equivalence classes of Financial Accounting Matrices.

Strictly speaking, in the proof above we have used, without proving it, that the matrix B would be invertible (that is, non-singular) if $\{v_1, \ldots, v_n\}$ are a basis of V. In fact, it is sufficient that they be linearly independent - but pursuing this thread too far would result in trying to write a crash course in linear algebra, a fate we'd like to avoid.